THE STABILITY OF PERIODIC MOTIONS IN THE CASE OF TWO ZERO ROOTS

(OB USTOICHIVOSTI PERIODICHESKIKH DVIZHENII V SLUCHAE DVUKH NULEVYKH KORNEI)

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S. V. KALININ
(Moscow)

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In his monograph [1], Kamenkov considered the stability of steady motions in the case of two zero roots. It is shown in the present paper that the method of Kamenkov may be extended, under certain conditions, to the stability investigation of periodic motions, also in the case of two zero roots.

1. Let us consider a system of differential equation of order $(n + 2)$, such that its characteristic equation has two zero roots and n roots with negative real parts. Systems of differential equations under this assumption are subdivided into two classes.

(a) Cases, in which zero roots correspond to one group of solutions; the system is reduced to the form

$$\frac{dx}{dt} = y + X(x, y, x_s), \qquad \frac{dy}{dt} = Y(x, y, x_s)$$
$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(x, y, x_s) \qquad (s=1,\ldots, n) \tag{1.1}$$

(b) Cases in which the two groups of solutions correspond to zero roots; the system is reduced to the form

$$\frac{dx}{dt} = X(x, y, x_s), \qquad \frac{dy}{dt} = Y(x, y, x_s)$$
$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(x, y, x_s) \qquad (s=1,\ldots, n) \tag{1.2}$$

We shall assume that in equations (1.1) and (1.2) the coefficients p_{sr} are bounded by periodic functions of t, with period ω, and are determined by the expressions

$$p_{sr} = c_{sr} + \varepsilon f_{sr}(t) \qquad \left(c_{sr} = \frac{1}{\omega}\int_0^\omega p_{sr}(t)\, dt\right) \tag{1.3}$$

where c_{sr} are constants, f_{sr} are periodic functions of t with period ω, and ϵ is a certain parameter.

The functions X, Y, X_s are holomorphic functions of x, y, x_s in the region $x_1^2 + \ldots + x_n^2 \leqslant A$ for $t \geqslant t_0$, and their expansion begins at least with terms of second power, such that the expansion coefficients of these functions are also bound periodic functions of t, of the same period and form as the coefficients p_{sr}.

2. Let us first consider the problem of stability, when it is reduced to the study of the system of equations (1.1). In this system the terms Y and X_s have the following expansion.

$$Y = (a^{(0)} + \varepsilon A^{(0)})\, x^{\alpha_0} + \ldots + (a^{(1)} + \varepsilon A^{(1)})\, x^{\alpha_1} + \ldots + \sum_k Q^{(k,\,0)} x^k + y \sum_k Q^{(k,\,1)} x^k + \ldots \tag{2.1}$$

$$X_s = (a_s^{(0)} + \varepsilon A_s^{(0)})\, x^{\beta_0} + \ldots + (a_s^{(1)} + \varepsilon A_s^{(1)})\, x^{\beta_1} + \ldots + X_s^*(x,\ y,\ x_s) \quad (s = 1, \ldots, n)$$

By means of transformations we may find that the power a_k of the critical variable x in the transformation Y may not be higher than the power β_k of the same variable in the corresponding terms of the expansion X_s, i.e. always $a_k \leqslant \beta_k$. In expansions $a^{(r)}$ and $a_s^{(r)}$ the constant quantities $A^{(r)}$ and $A_s^{(r)}$ are bounded periodic functions of t with period ω, while $Q^{(k,\,i)}$ are holonomic functions in variables x_s. In system (1.1) and in the expansion terms of the equations let us replace the periodic coefficients by constant coefficients, averaged over a period.

After this, the system (1.1) takes on the form studied by Kamenkov:

$$\frac{dx}{dt} = y + \overline{X}, \qquad \frac{dy}{dt} = \overline{Y}, \qquad \frac{dx_s}{dt} = c_{s1}x_1 + \ldots + c_{sn}x_n + \overline{X}_s \tag{2.2}$$

Expansions (2.1) are expressed as by Kamenkov in his paper [1].

The first equation of the system (2.2) will be transformed to the form

$$\frac{dx}{dt} = y^* \tag{2.3}$$

We shall also assume that the functions $x_s(x,\ y)$ are solutions of the equations

$$c_{s1}x_1 + \ldots + c_{sn}x_n + \overline{X}(x,\ y,\ x_s) = 0 \qquad (s = 1, \ldots, n)$$

The investigation of stability or instability of motion described by the system (2.2) is determined with the aid of the function of Liapunov or Chetaev. Let us for example consider the Chetaev function corresponding to system (2.2):

$$V = xy + W(x_1, \ldots, x_n) \tag{2.4}$$

where the function W is determined from the equation

$$\sum_{s=1}^{n} \frac{\partial W}{\partial x_s}(c_{s1}x_1 + \ldots + c_{sn}x_n) = x_1^2 + \ldots + x_n^2$$

Since all the roots of the equation have only negative real parts, the function W will be a sign definite form in the variables $x_1, \ldots, x_n$, and the function V will be positive-definite.

The total differential V' of the Chetaev function (2.4), by virtue of the original equations (1.1), will be of the form:

$$V' = y^2 + a^{(0)}yx^{\alpha_0+1} + a^{(1)}yx^{\alpha_1+1} + x_1^2 + \ldots + x_n^2 + \\ + \varepsilon H(A^{(0)}yx^{\alpha_0+1}, A^{(1)}yx^{\alpha_1+1}, \ldots, A_s^{(0)}x^{\beta_0}y, \ldots) + [R] \tag{2.5}$$

In the region $x > 0$ and $y > 0$, in which the inequality $V' > 0$ holds, the terms $a^{(0)}yx^{a_0+1}$ and $a^{(1)}yx^{a_1-1}$ will be positive if $a^{(0)} > 0$ and $a^{(1)} > 0$. H indicates some function of the periodic part of the coefficients; it becomes equal to zero together with ϵ.

The function $[R]$ contains higher order variables.

Since the function V' depends on time, it is sufficient to find a positive-definite time-independent function W^0, in order to make it sign-determined, such that one of the two expressions $(V' - W^0)$ or $(-V' - W^0)$ represents a positive function.

We set

$$W^\circ = \mu(y^2 + x_1^2 + \ldots + x_n^2) \tag{2.6}$$

and construct the expression $V' - \mu(y^2 + x_1^2 + \ldots + x_n^2)$, where μ is a certain positive number smaller than unity.

For the quadratic part of the expression (2.5) we write the determinant $D(\mu)$ in abbreviated form as

$$D(\mu) = \begin{vmatrix} 1-\mu+w_{00} & w_{01} \ldots & w_{0n} \\ w_{10} & 1-\mu+w_{11} \ldots & w_{1n} \\ \ldots & \ldots & \ldots \\ w_{n0} & w_{n1} \ldots & 1-\mu+w_{nn} \end{vmatrix} = 0 \tag{2.7}$$

The principal diagonal minors of the order q of the determinant $D(\mu)$ are polynomials in ϵ, with the free term $(1 - \mu)^q$ and with a part which depends on ϵ and which vanishes together with ϵ.

We require that the principal diagonal minors of the determinant (2.7) be positive, and as a consequence the function V' will be positive-definite if ϵ is not larger than unity, to be determined from Sylvester's condition. In this case the steady motion will be unstable.

A series of cases of stability of motion, described by system (1.1), will be investigated with the aid of Liapunov's function and the analysis is carried out as indicated above.

3. Let us now consider a stability problem when the system possesses two groups of solutions and is reduced to the form (1.2). The system of equations (1.2) is re-written in expanded form:

$$\frac{dx}{dt} = X(x, y) + \sum Q^{(k_1, k_2)} x^{k_1} y^{k_2}, \quad \frac{dy}{dt} = Y(x, y) + \sum P^{(k_1, k_2)} x^{k_1} y^{k_2}$$

$$\frac{dx_s}{dt} = p_{s1} x_1 + \ldots + p_{sn} x_n + \overline{X}_s(x, y) + \sum P_s^{(n_1, n_2)} x^{n_1} y^{n_2} \qquad (s = 1, \ldots, n) \quad (3.1)$$

Here all sums are extended over all integral positive numbers k_1, k_2, n_1 and n_2, the functions $Q^{(k_1, k_2)}$, $P^{(k_1, k_2)}$ and $P_s^{(n_1, n_2)}$ are holomorphic and their expansion begins with terms not higher than of the second order.

The quantities $X(x, y)$, $Y(x, y)$ and $X_s(x, y)$ in equations (3.1) are holomorphic functions in the variables x, y, not containing terms of the first order in their expansions; they begin with terms of order m and the expansions terminate with terms of order N. They are of the following form:

$$X(x, y) = (a^{(0)} + \varepsilon A^{(0)}) x^m + (a^{(1)} + \varepsilon A^{(1)}) x^{m-1} y + \ldots + X^{(m+N)}(x, y) + \ldots$$

$$Y(x, y) = (b^{(0)} + \varepsilon B^{(0)}) y^m + (b^{(1)} + \varepsilon B^{(1)}) y^{m-1} x + \ldots + Y^{(m+N)}(x, y) + \ldots$$

$$X_s(x, y) = X_s^{(m+N+1)}(x, y) + X_s^{(m+N+2)}(x, y) + \ldots \quad (3.2)$$

$$(k_1 + k_2 > m + N, \ s = 1, \ldots, n)$$

In the system (3.1) all coefficients p_{sr} and coefficients of expansions of the functions X, Y, X_s are bounded periodic functions of t, of the same period and form as was given earlier by the expression (1.3); in expansions (3.2) the quantities $a^{(r)}$ and $b^{(r)}$ are constants, $A^{(r)}$ and $B^{(r)}$ are periodic functions of t with period ω, ϵ is a certain parameter, N is a certain positive integer.

In system (3.1) and in expansions (3.2) we replace the periodic coefficients by constant coefficients, averaged over a period. The system (3.1) as well as the expansions of the terms of this system, will then be given by the same expressions as those indicated by Kamenkov in his paper [1].

The investigation of the system is carried out in accordance with the method developed by Kamenkov [1] with the aid of Liapunov's or Chetaev's function.

In all the cases, after the construction of the Chetaev or Liapunov function, the total derivative with respect to time t is determined on the basis of the original equations with periodic coefficients (3.1). To make V sign-determined, a positive-definite function $\overline{W}$ of the type (2.6) is sought which does not depend on t; the expression $(V' - \overline{W})$ or $(-V' - \overline{W})$ is constructed. In the quadratic part

$$V' - \mu(x^2 + y^2 + x_1^2 + \ldots + x_n^2) \qquad (0 < \mu < 1)$$

the determinant $D(\mu)$ of the type (2.7) is constructed. To make the function V' sign-determined it is necessary that all diagonal minors of the determinant be positive; in this manner, necessary limitations on the parameter ϵ are imposed. As long as ϵ does not exceed the value determined from Sylvester's conditions in studying stability of periodic motions in the case of two zero roots, the theorems of Kamenkov remain valid.

BIBLIOGRAPHY

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